Enumerating nets

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How many nets does a given polyhedron have?

A familiar result is that there are eleven nets for the cube (figure 1), but a proof of that fact is less well known. One aim of this article is to outline one method of establishing this result (which will also explain the ordering of the nets in the figure).
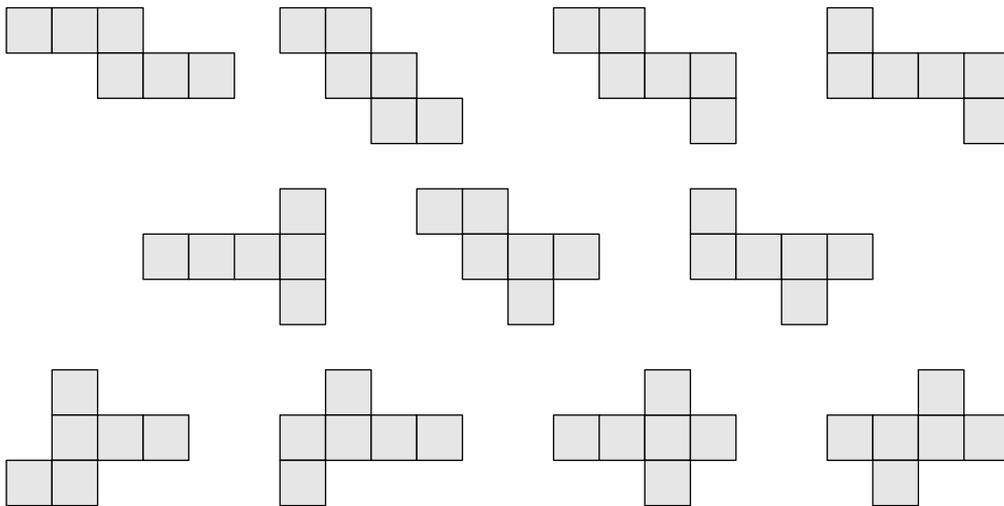

Figure 1

The method we describe can be applied to other polyhedra, but becomes less and less practical as the number of faces of the polyhedron grows. The method is essentially that given by Turney in [4].

Firstly, let us make clear exactly what we mean by a net, and what it means to say two nets are different.

Polyhedral nets

We only consider polyhedra without holes; indeed, we only consider *convex* poyhedra, meaning that the internal angle between two faces meeting at an edge is always less than 180°.

A *net* is an unfolding of the surface of a polyhedron produced by cutting the polyhedron along some of its edges and flattening it to a single, non-overlapping piece in the plane. One could cut out a net drawn on paper, crease along edges, and fold to make the three-dimensional polyhedron.

Note that some unfoldings may not give a net, since faces may overlap, a possibility that thankfully does not occur for the polyhedra we consider. Note also that some nets may be folded into more than one polyhedron, so that, in general, a net should specify which edges are to be joined. Once again, for the polyhedra we consider, this possibility does not occur: the folding procedure can

only be done one way, so edges need not be labelled. Both these issues are discussed in [1], for example.

Two nets are considered to be the same if one of them can be moved, possibly scaled, and perhaps turned over, so that it is exactly superimposed on the other.

The method

Consider graphs in which each node corresponds to a face of the polyhedron, and an edge of the graph corresponds to two connected faces. In this context an unfolding of the polyhedron corresponds to a tree. We start, therefore, with the complete set of all trees with n nodes, where n is the number of faces. (For the number of trees for small n see [3]; for the trees themselves see [2].)

Now we impose conditions on each tree using the combinatorial properties of the polyhedron; one way to do this is to mark the nodes of the tree in some way, for example by colouring them. The resulting set of marked trees will correspond to the set of nets of the polyhedron, so that instead of enumerating nets we may enumerate the marked trees.

Of course, it is necessary to prove that each different unfolding corresponds to a single marked tree, and that each such tree corresponds to a single unfolding; in other words, that there is a one-to-one mapping from the set of unfolded polyhedra to the set of marked n-node trees.

Examples

We apply the method to five polyhedra: the regular tetrahedron; a different triangular pyramid (figure 2); a triangular prism (figure 3); a square pyramid (figure 4); and the cube.
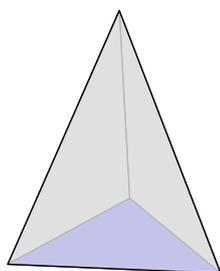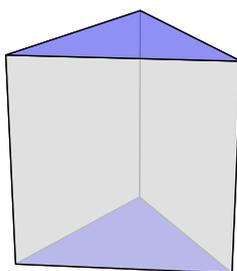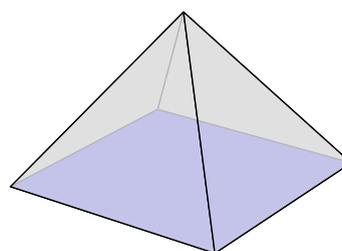

Figure 2 Figure 3 Figure 4

For each example, we explain how the trees should be marked. Note that the actual colours used are not important, so that two marked trees are not considered to be different if interchanging colours changes one tree into the other.

As stated above, we ought to prove for each example that the mapping from the set of unfolded polyhedra to the set of marked trees is one-to-one.

Proving that an unfolding leads to a single marked tree is usually straightforward: since the (unmarked) tree is determined by the way in which the faces in the net are joined, one only needs to check that an unfolding leads to a single marked tree. This will be the case provided the

markings are well-chosen. For example, opposite faces of the polyhedron are never adjacent in an unfolding, so correspond to non-adjacent nodes.

Proving the other direction—that a marked tree corresponds to a single unfolding—is a little more tricky. We indicate one way in which this may be done in the discussion of the square pyramid. A detailed proof for the cube is given in [4].

The regular tetrahedron

The regular tetrahedron has four faces, and there are just two trees with four nodes, which correspond to two nets (see figure 5).
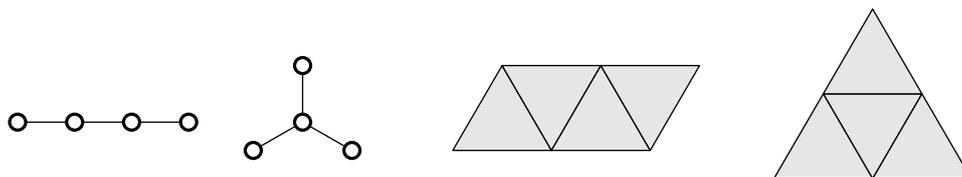
Figure 5

Another triangular pyramid

Suppose instead that the tetrahedron is not regular, but has one equilateral face and three isosceles, non-equilateral, faces, as shown in figure 2.

We may colour one node of the tree, corresponding to the equilateral face, leaving the other nodes uncoloured (since the other faces are indistinguishable from one another). This may be done in two different ways for each of the two trees in figure 5. The process leads to the four coloured trees shown in figure 6. The trees correspond to the four nets shown in figure 7.
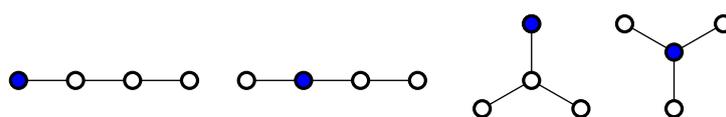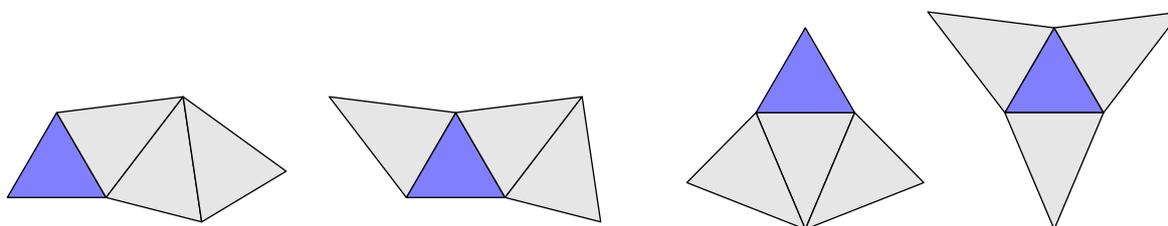

Figure 6

Figure 7

3

A triangular prism

Consider the triangular prism with regular faces, shown in figure 3, where the triangular faces are coloured blue.

The prism has five faces, so we start with the trees with five nodes—there are just three of them, shown in figure 8.
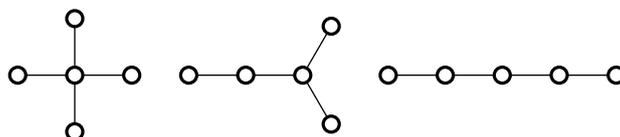

Figure 8

Now the two triangular faces of the prism are opposite, and so will never be adjacent in an unfolding. Thus we colour exactly two non-adjacent nodes, leaving the other nodes uncoloured. This may be done in one way for the left-hand tree in figure 8, in four ways for the middle tree (shown in figure 9), and in four ways for the right-hand tree.
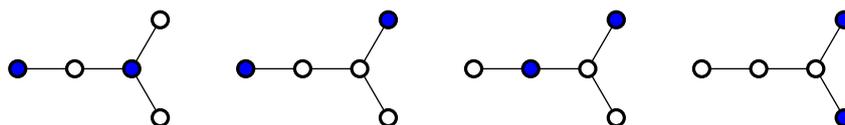

Figure 9

Each marked tree corresponds to a different net, so there are nine possible nets, which we leave the reader to find.

A square pyramid

Consider the pyramid shown in figure 4, with a square base and four equilateral slant faces.

The polyhedron has five faces, so once again we start with the set of three five-node trees shown in figure 8.

For this polyhedron, not only is the base different to the other faces, but the four triangles may be divided into two opposite pairs. So now we mark the trees by colouring one node in one colour (corresponding to the base), a non-adjacent pair of nodes in a second colour (corresponding to one pair of opposite faces), and another non-adjacent pair of nodes in a third colour. For example, the four marked trees in figure 10 result from the right-hand tree in figure 8.

Figure 10

In all there are eight possible marked trees, leading to eight nets. The details are left to the reader.

We shall use this example to indicate one possible proof that a marked tree leads to a single net, by demonstrating that the tree in figure 11 determines the net in figure 12.
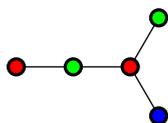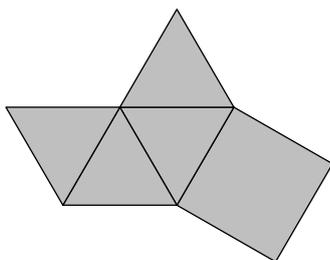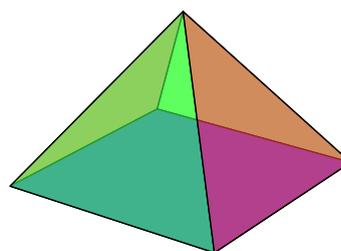
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Figure 11   Figure 12   Figure 13

Firstly, we colour the polyhedron (figure 13) using the same rules that we used to colour the nodes of the tree: use one colour for the base; a second colour for one pair of opposite triangular faces; and a third colour for the other pair. Here we have used blue, red and green, but the actual colours are not relevant.

Figure 14 shows the five coloured faces of the polyhedron. In the figure, each edge has also been marked with a coloured dot to indicate the colour of the face which joins that edge. Since the apex of the pyramid is different to the other four vertices, we identify it on each triangular face with a black dot. Note that the marked triangles are identical in pairs.
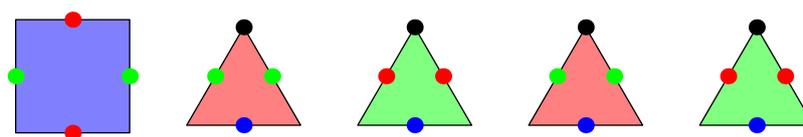

Figure 14

Now the marked tree indicates which faces should be joined, so we use it as a "blueprint" to assemble the faces. For example, the leftmost edge of the tree indicates that a red face is joined to a green face, so we place a red triangle and a green triangle together. Continuing in this way we obtain figure 15, where the grey lines connect pairs of faces that are to be joined.
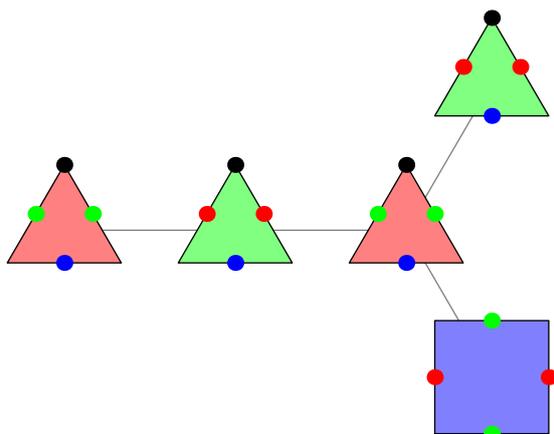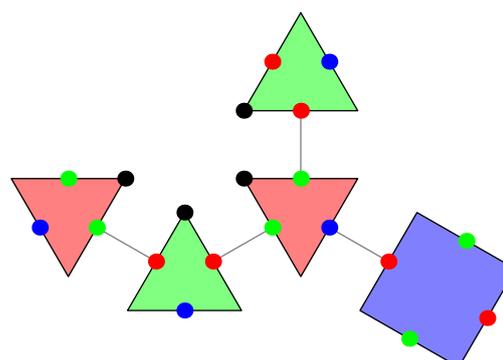

Figure 15   Figure 16

Next we rotate the faces so that the edges "match"—the colour of the dot matches the colour of the adjacent face—and the black dots correspond, as shown in figure 16. Finally, we join the faces together, leading to the net shown in figure 12.

The reader is encouraged to use this process to check that each of the other marked trees maps to a single unfolding.

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The cube

A cube has six faces, so we start with the set of six trees with six nodes, shown in figure 17.
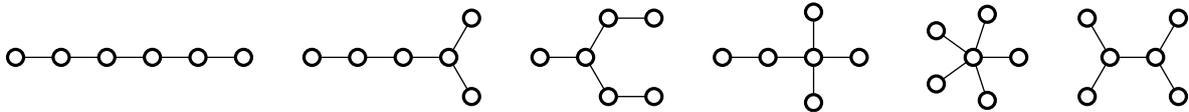

Figure 17

Opposite faces of the cube are never adjacent, so non-adjacent nodes of a tree resulting from an unfolding can be paired. Thus we mark the trees using three colours, one for each pair of non-adjacent nodes.

Figure 18 shows the eleven ways that the trees in figure 17 can be coloured in this way. Note that it is impossible to colour the penultimate tree in figure 17 in the required way since the central node is adjacent to all the others.
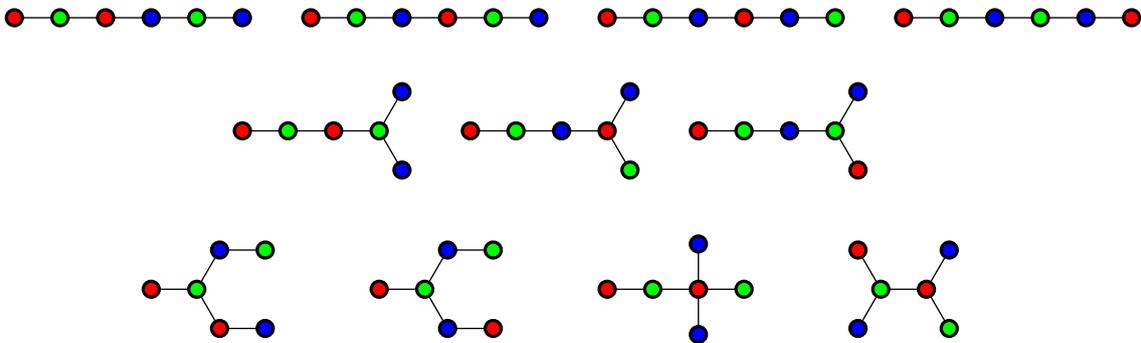

Figure 18

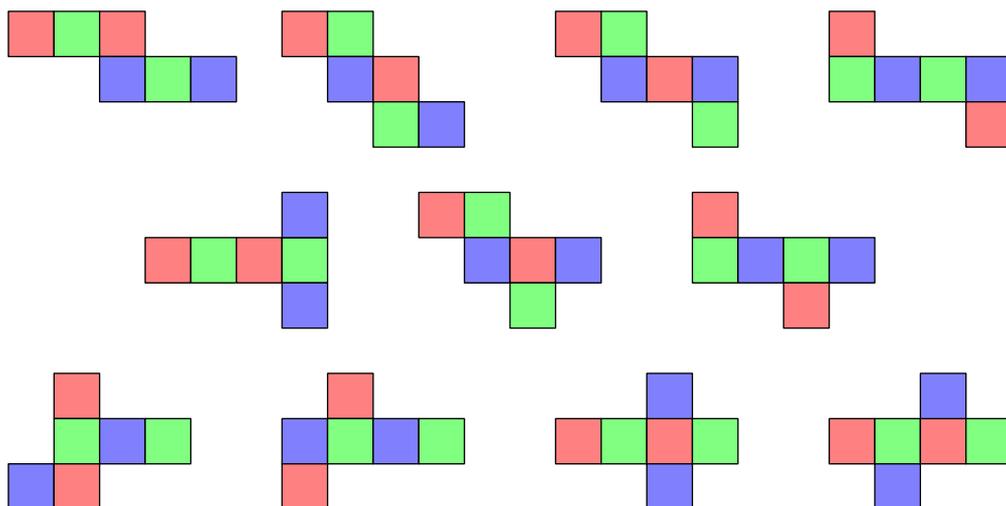The marked trees in figure 18 correspond to the eleven nets shown in figure 19.

Figure 19

6

References

[1]   Joseph Malkevitch. *Nets: A Tool for Representing Polyhedra in Two Dimensions*. Feature Column from the AMS. Feb. 2002.
URL: http://www.ams.org/samplings/feature-column/fcarc-nets.

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[4]   Peter Turney. 'Unfolding the Tesseract'.
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